INTEGRATION OF RICCATI'S EQUATION.

By MR. LEVI W. MEECH, Norwich, Conn.

Since the publication of my paper on the Integration of Riccati's Equation * it has been found that the series may take the form of a *definite integral.* A practical application occurs in the problem of mechanics, in tracing the path of a projectile in a resisting medium with friction as the square of the velocity. By a well-known integral, and by multiplying both numerator and denominator of the common binomial series by equal factors, which does not alter the value, we have

$$\int_0^\infty y^m \varepsilon^{-y}\, dy = 1 \,.\, 2 \,.\, 3 \,.\; .\; .\; m = m\,!,$$

$$(1+b)^m = \frac{m\,!}{m\,!} + \frac{m\,!\,b}{(m-1)\,!} + \frac{m\,!\,b^2}{1\,.\,2\,(m-2)\,!} + \frac{m\,!\,b^3}{1\,.\,2\,.\,3\,(m-3)\,!} + \;.\;.\;.$$

Hence in the last formula (18),† where b and c have *like* signs, by making

$$2q'z = \frac{4}{n}\sqrt{(bcx^n)} = h$$

and

$$\tfrac{1}{2}h + C' = t,$$

if

$$\frac{du}{dx} + bu^2 = cx^m,$$

$$N = \int_0^\infty \frac{y^i\,\varepsilon^{-y}}{i\,!}\left\{\left(1+\frac{y}{h}\right)^i \varepsilon^{-t} + \left(1-\frac{y}{h}\right)^i \varepsilon^{+t}\right\} dy,$$

$$\varDelta = \int_0^\infty \frac{y^{i-1}\,\varepsilon^{-y}}{(i-1)\,!}\left\{\left(1+\frac{y}{h}\right)^{i-1} \varepsilon^{-t} - \left(1-\frac{y}{h}\right)^{i-1} \varepsilon^{+t}\right\} dy,$$

$$u = -\frac{N}{\varDelta}\cdot\sqrt{\left(\frac{cx^m}{b}\right)}\,. \qquad (19)$$

By changing the variable from y to θ, making

$$y = \tan\theta, \quad dy = \frac{d\theta}{\cos^2\theta},$$

to facilitate quadrature, the limits of the definite integral become $\frac{1}{2}\pi$ and o. the common process, the quadrant is divided into three or more equal parts, N, $\cos^2\theta$, and $\varDelta$ are computed for $\theta = 15°, 45°, 75°$, or for the middle of e

* *Annals of Mathematics*, Vol. I. pp. 97–103.

† Ibid, p. 102.

part. Since $i\,!$ in N, divided by $(i-1)\,!$ in Δ, gives simply i, the example where i is not an integer will occasion no difficulty, and the quadrature converges rapidly.

When b and c have *unlike* signs, the factor $\sqrt{(-1)}$ occurring in h, and in $\sqrt{\left(\frac{cx^m}{b}\right)}$, will evidently change the sum and difference of ε^{-t} and ε^{t} into cos t and sin t, requiring a separate formula. It should be noted that on p. 102 a few errors in algebraic signs may be corrected from the preceding page, or from the definite integral (19). If, in verification of Δ, we add 1 to i, the two parts of Δ evidently become identical with N, except the connecting sign. Also when $i=0$, and $m=0$, the result of (18) or (19) has the same value, with only a change in form, as equation (2), where the arbitrary constant

$$C = -\varepsilon^{-2C'}, \quad \text{and} \quad u = -\sqrt{\frac{c}{b}} \cdot \frac{\varepsilon^{-t}+\varepsilon^{t}}{\varepsilon^{-t}-\varepsilon^{t}} \cdot \frac{\varepsilon^{-t}}{\varepsilon^{-t}}.$$

In integrating by continued fractions when i is an integer, the formulæ (20) and (21) are here materially simplified from p. 99 of Vol. I, compared with Boole's *Differential Equations.* The first of the two forms of the continued fraction is found from Riccati's Equation by assuming and substituting successively from $\frac{du}{dx}+bx^2=cx^m$,

$$m+2=n, \quad \sqrt{(cb)}=k, \quad u=\frac{ky}{bx};$$

$$y=\frac{1}{k}+\frac{x^n}{y_1},$$

$$y_1=\frac{1+n}{k}+\frac{x^n}{y_2},$$

$$y_2=\frac{1+2n}{k}+\frac{x^n}{y_3},$$

$$\cdots$$

$$y_{i-1}=\frac{1+(i-1)\,n}{k}+\frac{x^n}{y_i},$$

$$\frac{xdy_i}{dx}-(1+in)\,y_i+ky_i^2=kx^n.$$

When i is a positive integer, $1+in=\frac{1}{2}n$, and the last equation gives the exact value of y_i through (2) or (3), by first making $x^n=x_1^2$, and $y_i=zx_1$; whence

$$\frac{dz}{dx_1}+\frac{2}{n}kz^2=\frac{2}{n}k.$$

Then writing out the two-fold values of $y, y_1, \ldots y_i$, reducing, and making $k^2x^n = t$, we have

$$bx \cdot u = 1 + \frac{t}{1+n+} \frac{t}{1+2n+} \frac{t}{1+3n+} \cdots \frac{t}{1+(i-2)n+} \frac{t}{-\frac{1}{2}n+} \frac{t}{ky_i}. \tag{20}$$

This gives u for the series (5), where $n = \frac{-2}{2i-1}$. Again, for the series (6), or $n = \frac{2}{2i+1}$,

$$bx \cdot u = \frac{t}{n-1+} \frac{t}{2n-1+} \frac{t}{3n-1+} \cdots \frac{t}{(i-1)n-1+} \frac{t}{-\frac{1}{2}n+} \frac{t}{ky_i}, \tag{21}$$

where
$$ky_i = \sqrt{t} \cdot \frac{C\varepsilon^{\frac{4\sqrt{t}}{n}} + 1}{C\varepsilon^{\frac{4\sqrt{t}}{n}} - 1},$$

$$= \sqrt{-t} \cdot \cot\left(C + \frac{2\sqrt{-t}}{n}\right),$$

according as b and c have like or unlike signs. The simplified formula (21) differs apparently from the result of Prof. Boole, but the sum and the number of terms in both, is the same. For example, when $m + 2 = n = \frac{2}{9}$, or $i = 4$, the formula (21) becomes

$$bx \cdot u = \frac{t}{-\frac{7}{9}+} \frac{t}{-\frac{5}{9}+} \frac{t}{-\frac{3}{9}+} \frac{t}{-\frac{1}{9}+} \frac{t}{ky_i}.$$

When i is not an integer, let i' denote its value. When i' differs but little from an integer, we may suppose the series $1+n$, $1+2n, \ldots$ running forward from the beginning, and another series $\ldots, -\frac{3}{2}n, -\frac{1}{2}n$, running back from the known integral ky_i, to intersect near $-\frac{1}{2}n$. A sufficient approximation may often be obtained by substituting half the sum of $1+(i-2)n$ and $-\frac{1}{2}n$ in the place of $-\frac{1}{2}n$ in the last denominator but one of (20); or half the sum of $in - 1$ and $-\frac{1}{2}n$ in place of $-\frac{1}{2}n$ in (21), without other change of the formulæ. For other values of i' the possibility of connecting the two series is offered by some other less simple method of interpolation.